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Unanimity under Ambiguity

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Abstract

Ellis (2016) introduced a variant of the classic (jury) voting game in which voters have ambiguous prior beliefs. He focussed on voting under majority rule and the implications of ambiguity for Condorcet's Theorem. Ryan (2021) studied Ellis's game when voting takes place under the unanimity rule. His focus was on the implications of ambiguity for the "jury paradox" (Feddersen and Pesendorfer, 1998). Neither paper described all equilibria of these games, though both authors identified equilibria with a very different structure to those in the respective games without ambiguity. We complete the description of all equilibria of voting games under the unanimity rule. In particular, we identify equilibria having the same form as those in Feddersen and Pesendorfer (1998), as well as equilibria with a "dual" form.

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1 Introduction

This paper considers a binary decision to be made by a committee – canonically, a jury – through a voting procedure. The jury must vote on whether to convict or acquit a defendant, based on whether they believe the defendant to be guilty or innocent.

Recall the classical model of this situation, which formalises the famous analysis of Condorcet. All jurors agree on the “correct” decision conditional on the unobserved state – convict if guilty, acquit if innocent – and all share a common prior probability on guilt. Jurors receive independent, private signals before casting their votes. Each juror may receive a “guilty” signal or an “innocent” signal, with the eponymous signal being more likely in either state. All jurors wish to maximise the probability of a correct decision and there is no communication between jurors after they receive their private information. The equilibria of these classical voting games have been analysed by (*inter alia*) Austen-Smith and Banks (1996) and McLennan (1998).

Non-classical versions of the model that include various forms of communication have also been analysed: for example, by Coughlan (2000) and Gerardi and Yariv (2007). Both the classical (without communication) and non-classical (with communication) versions have been studied experimentally, and many features of the observed behaviour accord remarkably closely with the theoretical predictions: see, for example, Guarnaschelli, McKelvey and Palfrey (2000) and Goeree and Yariv (2011).

Ellis (2016) studies another variation on the classical voting model in which the common prior is ambiguous – jurors share a *set* of prior probability distributions over the states. In Ellis’s model, jurors aim to maximise the *minimised (over priors)* probability of a correct decision. Ellis focusses exclusively on the majority rule since his objective is to establish a generalisation of Condorcet’s Jury Theorem (Young, 1988).

Ryan (2021) considers a variant of Ellis’s model in which losses may be asymmetric – jurors may suffer a higher utility penalty from convicting the innocent than acquitting the guilty – and conviction requires unanimity rather than a majority of guilty votes. This variant essentially adds prior ambiguity to the games studied by Feddersen and Pesendorfer’s (1998). For the purposes of this Introduction, let us call them *ambiguous FP games*. The objective of Ryan (2021) is to generalise Feddersen and Pesendorfer’s (1998) Jury Paradox to ambiguous FP games – to show that the probability of convicting the innocent remains bounded away from zero as the jury size increases.

In Feddersen and Pesendorfer’s (1998) model, symmetric equilibria of generic voting games have a simple structure. There is always a “trivial” equilibrium in which all jurors vote to acquit irrespective of their signals. Under a mild restriction there is also a unique non-trivial equilibrium. In this equilibrium, jurors always vote to convict when they receive a guilty signal but may also vote to convict with positive probability (less than 1) when

they receive an innocent signal.¹ The logic underpinning this equilibrium is not difficult to understand. Optimal responses can be determined by conditioning on the event that one’s vote is pivotal. Under the unanimity rule, pivotality means that all other jurors vote to convict. This may convey overwhelming evidence of guilt, causing jurors to ignore innocent signals unless guilty votes are noisy indicators of guilty signals. The larger the jury, the noisier these indicators need to be in order to sustain the informativeness of voting, so the higher the equilibrium probability of voting against an innocent signal.

Ryan (2021) shows that ambiguity introduces a richer array of equilibrium behaviours, even for large juries. In some games voters randomise after *either* signal, and they may do so *responsively* (i.e., different signals induce different randomisations) or *non-responsively*. Ryan (2021) provides a detailed analysis of all non-responsive equilibria of ambiguous FP games, as well as the “strictly mixed” responsive equilibria but leaves open the possibility that additional equilibria may exist for some games. In this paper we complete the description of all equilibria of ambiguous FP games.

Two new classes of equilibria are analysed: one with the structure of Feddersen and Pesendorfer’s (1998) responsive equilibria and another with a “dual” structure in which jurors vote to acquit when they receive an innocent signal but randomise when they receive a guilty signal. We characterise both types of equilibrium and provide necessary and sufficient conditions for their existence. Combining the results presented here with those in Ryan (2021), it is possible to identify *all* equilibria of *any* ambiguous FP game.

2 The model

2.1 Voting problems

The model that we analyse is precisely that of Ryan (2021), which in turn is a hybrid of Feddersen and Pesendorfer (1998) and Ellis (2016), mostly adopting the notation of the latter. The model is described in detail in Section 2 of Ryan (2021); we only provide a brief summary here. Readers familiar with the companion paper may skip this section without loss of continuity.

There is a set $I = \{1, 2, \dots, N + 1\}$ of jurors, with generic member i , which makes a decision $d \in D = \{A, B\}$ by secret ballot. We interpret A as the decision to “acquit” the defendant; hence B corresponds to entering a conviction.² We use the same notation for decisions and votes: each juror may vote A for acquittal (the “innocent” vote) or B for conviction (the “guilty” vote). The outcome is determined by the unanimity rule: the defendant is acquitted – decision $d = A$ is made – unless *all* jurors vote for conviction, in which case decision $d = B$ is made.

¹The restriction is that voters strictly prefer to convict [respectively, acquit] conditional on knowing that all [respectively, no] jurors received guilty signals.

²Our notation follows Ellis (2016) to facilitate comparison with his analysis.

The defendant may be innocent or guilty, represented by the state $s \in S = \{a, b\}$, where $s = a$ is the state of innocence and $s = b$ the state of guilt. (Think of b as the state in which the defendant is “bad”.) Jurors share common ambiguous prior information about s . The prior probability of $s = a$ is objectively known to lie in the interval $[\underline{p}, \bar{p}] \subseteq (0, 1)$ but nothing more than this. Prior to casting their vote, each juror receives a private signal $t \in T = \{1, 2\}$. Conditional on $s \in S$, these signals are independently and identically distributed with $\Pr(1|a) = \Pr(2|b) = r \in (\frac{1}{2}, 1)$.

Let $\Omega = S \times T^I$ denote the state space characterising all *ex ante* uncertainty. Together with r , each $p \in [\underline{p}, \bar{p}]$ determines a probability over Ω . The (closed and convex) set of probabilities over Ω determined by $[\underline{p}, \bar{p}]$ is denoted by Π . After receiving their signal, a juror uses the *full Bayesian updating (FBU)* rule to update their beliefs: they update each element of Π using Bayes’ Rule to obtain a set of posterior probabilities on Ω (Fagin and Halpern, 1990; Jaffray, 1992). The posterior interval for the conditional probability $\Pr(a|t_i = t)$ is independent of i and denoted by $\Pi_t = [\underline{\pi}_t, \bar{\pi}_t]$, with generic element π_t . Since $[\underline{p}, \bar{p}] \subseteq (0, 1)$ it follows that $\Pi_t \subseteq (0, 1)$.

Voters share a common utility function, $u : D \times S \rightarrow \mathbb{R}$, with $u(A, a) = u(B, b) = 1$, $u(A, b) = 0$ and $u(B, a) = -c$, where $c \geq 0$. Thus, A is the “correct” decision in state a and B is the “correct” decision in state b . Ellis’s model (or rather, a two-state special case of his model) is obtained by setting $c = 0$. When $c > 0$ convicting the innocent results in lower utility than acquitting the guilty.

Note that:

$$\begin{aligned} \pi u(B, a) + (1 - \pi) u(B, b) &\geq \pi u(A, a) + (1 - \pi) u(A, b) \\ \Leftrightarrow \pi &\leq \frac{1}{2 + c}. \end{aligned}$$

The quantity

$$1 - \left(\frac{1}{2 + c} \right) = \frac{1 + c}{2 + c}$$

is what Feddersen and Pesendorfer (1998) refer to as the “threshold of reasonable doubt”; it is the minimum probability of guilt ($s = g$) necessary to justify the decision to convict. In the absence of ambiguity, it is therefore optimal to vote for conviction iff the juror’s posterior probability on $s = g$, after incorporating their private information *and the implications of pivotality*, weakly exceeds this threshold. As noted in Ellis (2016),³ in the presence of ambiguity we can no longer condition on pivotality when determining optimal voting behaviour. We return to this point below.

A *voting problem* is a vector $V = (N, c, r, \underline{p}, \bar{p})$, with $N \in \{1, 2, \dots\}$, $c \geq 0$, $r \in (\frac{1}{2}, 1)$ and $0 < \underline{p} \leq \bar{p} < 1$. The set of all voting problems is denoted by \mathcal{V} .

³And elaborated in Pan (2019).

2.2 Strategies and equilibria

Each voting problem induces a voting game. Let σ_t^i denote the probability that $i \in I$ votes B after observing $t \in T$, and let $\sigma^i = (\sigma_1^i, \sigma_2^i)$ denote i 's strategy. We focus on *symmetric* profiles, in which each voter follows the same strategy, so we mostly omit the i superscript in what follows. We therefore abuse notation and refer to $\sigma = (\sigma_1, \sigma_2)$ interchangeably as the strategy of a generic voter in a symmetric profile or as the symmetric profile itself.

Consider a generic voter i who believes that each other voter follows the strategy $\sigma = (\sigma_1, \sigma_2)$. Following Ellis (2016) we let ρ_s denote the probability that voter i 's vote is pivotal, conditional on being in state $s \in S$; we let θ_s denote the probability that i is not pivotal *and* a correct decision is made, conditional on being in state $s \in S$. Since conviction requires unanimity, we have $\theta_a = 1 - \rho_a$, $\theta_b = 0$,

$$\rho_a = [r\sigma_1 + (1-r)\sigma_2]^N \quad (1)$$

and

$$\rho_b = [(1-r)\sigma_1 + r\sigma_2]^N \quad (2)$$

It is important to observe that this notation suppresses the dependence of ρ_a , θ_a and ρ_b on the symmetric strategy profile, σ .

After observing their private signal $t \in T$, voter i chooses σ_t^i according to the maxmin expected utility (MEU) rule. Hence:

$$\sigma_t^i \in \arg \max_{x \in [0,1]} \left[\min_{\pi_t \in \Pi_t} V(x, \sigma; \pi_t) \right] \quad (3)$$

where

$$\begin{aligned} V(x, \sigma; \pi_t) &= \pi_t [\rho_a (1 - x - cx) + \theta_a - (1 - \rho_a - \theta_a) c] + (1 - \pi_t) [\rho_b x + \theta_b] \\ &= \pi_t [\rho_a (1 - x - cx) + (1 - \rho_a)] + (1 - \pi_t) \rho_b x \end{aligned}$$

We use “equilibrium” as shorthand for a *symmetric Bayesian Nash equilibrium* of this game. Thus, $\sigma = (\sigma_1, \sigma_2)$ is an *equilibrium* iff σ_t^i satisfies (3) for each $t \in T$. The profile $\sigma = (0, 0)$ is an equilibrium of any voting problem, albeit a trivial one, since $\rho_a = \rho_b = 0$. We are interested in *non-trivial* equilibria.

Because the minimising posterior in (3) may vary with σ_t^i we can no longer condition on pivotality when determining best responses. Ryan (2021) derives the best response correspondence on the domain of non-trivial symmetric profiles. This is summarised by Figure 1, which reproduces Ryan (2021, Figure 1). In this figure,

$$\pi^*(\sigma) = \frac{\rho_b}{\rho_b + (1+c)\rho_a} = \frac{1}{1 + (1+c)(\rho_a/\rho_b)} \quad (4)$$

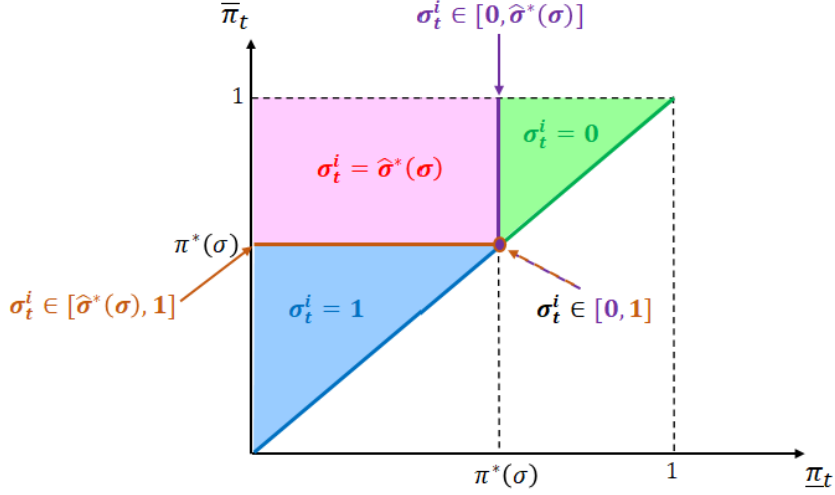


Figure 1: Optimal responses

and

$$\hat{\sigma}^*(\sigma) = \min \{ \sigma^*(\sigma), 1 \}$$

where

$$\sigma^*(\sigma) = \frac{1}{\rho_b + (1+c)\rho_a} = \frac{\pi^*(\sigma)}{\rho_b} \quad (5)$$

The figure is used to determine best responses as follows. Suppose voter i believes that each rival uses strategy $\sigma \neq (0, 0)$ and i has received signal $t \in T$. To identify i 's optimal vote, we use σ to calculate $\pi^*(\sigma)$ and $\hat{\sigma}^*(\sigma)$ and locate the point $(\underline{\pi}_t, \bar{\pi}_t)$ in Figure 1. Voter i 's optimal response (σ_t^i) is determined by the coloured region into which $(\underline{\pi}_t, \bar{\pi}_t)$ falls, as indicated in the figure. For example, if $(\underline{\pi}_t, \bar{\pi}_t)$ lies in the green region, then it is optimal to choose $\sigma_t^i = 0$ (i.e., to vote to acquit). Along the boundaries between the green, pink and blue regions, multiple optimal values for σ_t^i may exist. Within the pink region, excluding its boundary, there is a unique optimal value for σ_t^i but this value, $\hat{\sigma}^*(\sigma)$ may be strictly between 0 and 1. In this case, randomisation is necessary for an optimal response; with ambiguous priors, randomisation may be a valuable hedge against uncertainty.

Since $(\underline{\pi}_1, \bar{\pi}_1) \gg (\underline{\pi}_2, \bar{\pi}_2)$, the point $(\underline{\pi}_1, \bar{\pi}_1)$ will lie *strictly to the northeast* of the point $(\underline{\pi}_2, \bar{\pi}_2)$ when plotted in Figure 1. It follows that $\sigma_2 \geq \sigma_1$ in any equilibrium (Ryan, 2021, Lemma 4.1). In the terminology of Feddersen and Pesendorfer (1998), an equilibrium is *responsive* if $\sigma_2 > \sigma_1$ and *non-responsive* if $\sigma_1 = \sigma_2$. An equilibrium is *strictly mixed* if $0 < \sigma_1 \leq \sigma_2 < 1$.

Since $\sigma_1 \leq \sigma_2$ in any equilibrium, all non-trivial equilibria must fall into one of the following five categories:

C: Non-responsive with $\sigma = (1, 1)$ so all vote to convict.

MNR: Mixed non-responsive equilibria with $0 < \sigma_1 = \sigma_2 < 1$.

SMR: Strictly mixed responsive equilibria with $0 < \sigma_1 < \sigma_2 < 1$.

FP: Responsive equilibria with $0 \leq \sigma_1 < \sigma_2 = 1$.

DFP: “Dual” FP equilibria with $0 = \sigma_1 < \sigma_2 < 1$.

Ryan (2021) characterises all equilibria in the first three categories, and obtains necessary and sufficient conditions for their existence.⁴ We focus here on the last two. Ryan (2021, Lemma 4.3) shows that any voting problem has *at most one* FP equilibrium. This paper establishes necessary and sufficient conditions for the existence of an equilibrium in the FP class (Section 3.2). We also determine the value of σ_1 in such an equilibrium (Section 3.1). Equilibria of the DFP class have not been considered elsewhere. We obtain necessary and sufficient conditions for such equilibria to exist, and determine the value of σ_2 when they do (Section 4).

3 FP equilibria

Let $\Gamma^{\text{FP}} = \{(\sigma_1, 1) \mid 0 \leq \sigma_1 < 1\}$. For any voting game, at most one profile in the set Γ^{FP} can be an equilibrium. In this section we determine conditions on $V \in \mathcal{V}$ that are necessary and sufficient for Γ^{FP} to contain an equilibrium, and we identify the unique equilibrium element of Γ^{FP} for voting games that satisfy these conditions.

Given $\sigma \in \Gamma^{\text{FP}}$, the quantities $\hat{\sigma}^*(\sigma)$ and $\pi^*(\sigma)$ suffice to determine whether or not σ is an equilibrium. Let us therefore define the functions $g : [0, 1] \rightarrow \mathbb{R}_+$ and $h : [0, 1] \rightarrow \mathbb{R}_+$ as follows:

$$g(\sigma_1) \equiv \pi^*((\sigma_1, 1))$$

and

$$h(\sigma_1) \equiv \hat{\sigma}^*((\sigma_1, 1)).$$

Thus, $\hat{\sigma}^*((\sigma_1, 1)) = \min\{h(\sigma_1), 1\}$. We define these functions on $[0, 1]$ for convenience but only their values on $[0, 1)$ are relevant for assessing profiles in Γ^{FP} . The following result summarises the salient properties of these functions, and functions satisfying these properties are depicted in Figure 2.

Lemma 3.1 *The functions h and g are continuous and strictly decreasing, with $h(\sigma_1) > g(\sigma_1) > 0$ for all $\sigma_1 \in [0, 1)$ and $g(\sigma_1) < 1$ for all $\sigma_1 \in [0, 1]$. Function h has a unique fixed point, $\tilde{\sigma}_1$. Moreover, $\tilde{\sigma}_1 \in (0, 1)$ and*

$$\sigma_1 \geq h(\sigma_1) \quad \text{as} \quad \sigma_1 \geq \tilde{\sigma}_1 \tag{6}$$

⁴Proposition 4.2 in Ryan (2021) provides existence conditions for equilibria in the SMR class. They are implicitly characterised in the proof of Proposition 4.2.

If

$$(1 - r)^N + (1 + c)r^N > 1 \tag{7}$$

then $h(\sigma_1) \in (0, 1)$ for all $\sigma_1 \in [0, 1)$. Otherwise, there exists $\hat{\sigma}_1 \in [0, \tilde{\sigma}_1)$ such that

$$h(\sigma_1) \in (0, 1) \Leftrightarrow \sigma_1 > \hat{\sigma}_1.$$

Proof. From (1)-(2) and (5) we have

$$h(\sigma_1) = \frac{1}{[r\sigma_1 + (1 - r)]^N + (1 + c)[(1 - r)\sigma_1 + r]^N}.$$

and

$$g(\sigma_1) = h(\sigma_1)[(1 - r)\sigma_1 + r]^N$$

It is obvious that both functions are strictly positive, that $h(\sigma_1) \geq g(\sigma_1)$ with equality iff $\sigma_1 = 1$, and that $g < 1$. The denominator of $h(\sigma_1)$ is a continuous and strictly increasing function of σ_1 , so h is continuous and strictly decreasing. Since

$$[(1 - r)\sigma_1 + r]^N$$

is also a continuous and strictly decreasing function of σ_1 , function g shares these properties as well. Using

$$h(1) = (2 + c)^{-1} \in (0, 1)$$

it follows easily that h has a unique fixed point satisfying (6), and this point lies in $(0, 1)$. The remaining claims may be deduced from

$$h(0) = \frac{1}{(1 - r)^N + (1 + c)r^N}.$$

■

For the subsequent analysis it will be useful to define $\tilde{\pi} = g(\tilde{\sigma}_1)$ and $\hat{\pi} = g(\hat{\sigma}_1)$.

Figure 2 illustrates the facts asserted in Lemma 3.1 (without pretense to accuracy in any other details of the functions, except their values at $\sigma_1 \in \{0, 1\}$).⁵ Panel (a) exhibits a case satisfying (7) and Panel (b) a case where this condition fails. Note that (7) holds when $N = 1$ (unless $c = 0$) but must fail if N is sufficiently large. The quantities $\tilde{\sigma}_1$ and $\hat{\sigma}_1$ depend on r , c and N , though this dependence is suppressed in our notation.

It is important to note that when $\sigma \in \Gamma^{\text{FP}}$, the sign of $\sigma_1 - \hat{\sigma}^*(\sigma)$ depends on whether σ_1 is above or below the fixed point, $\tilde{\sigma}$.

Lemma 3.2 *Suppose $(\sigma_1, 1) \in \Gamma^{\text{FP}}$. Then $\sigma_1 \geq \tilde{\sigma}_1$ iff $\sigma_1 \geq \hat{\sigma}^*((\sigma_1, 1))$.*

⁵For example, they are drawn as if linear, but of course they are actually non-linear.

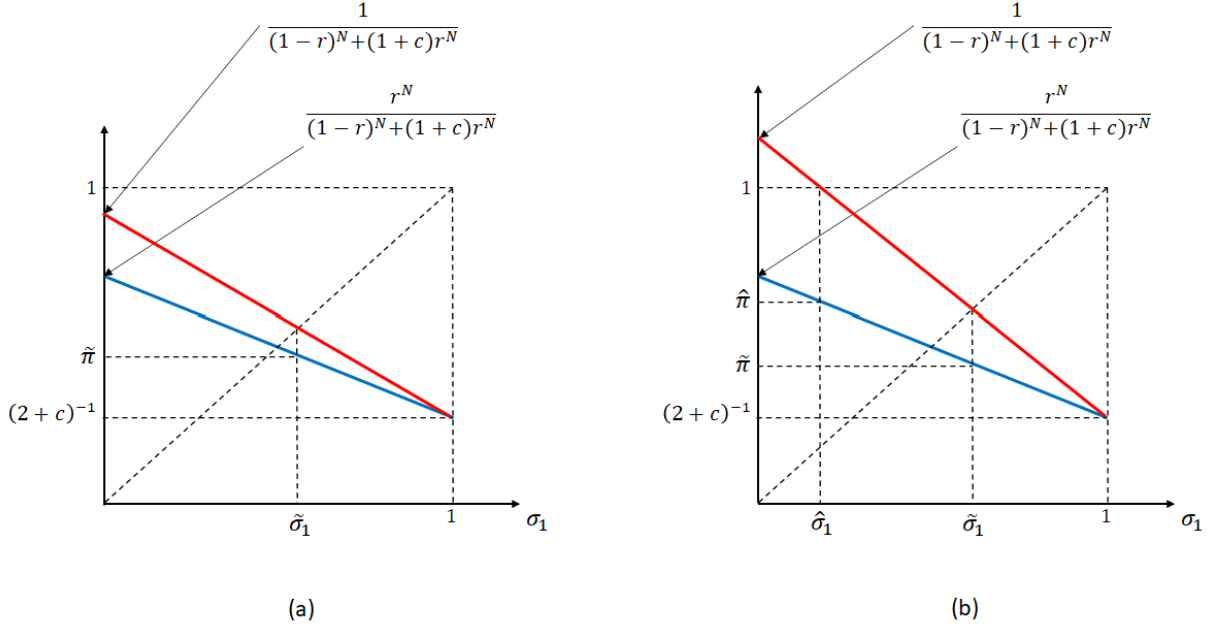


Figure 2: Functions g [blue] and h [red]. In Panel (a) condition (7) is satisfied; in Panel (b) it is not.

Proof. Since $(\sigma_1, 1) \in \Gamma^{\text{FP}}$ we have $\sigma_1 < 1$.

Suppose $\sigma_1 < \tilde{\sigma}_1$. Then $\sigma_1 < h(\sigma_1)$, so $\sigma_1 < \min\{h(\sigma_1), 1\} = \hat{\sigma}^*((\sigma_1, 1))$.

Suppose $\sigma_1 > \tilde{\sigma}_1$. Then $\sigma_1 > h(\sigma_1) \geq \min\{h(\sigma_1), 1\} = \hat{\sigma}^*((\sigma_1, 1))$.

Suppose $\sigma_1 = \tilde{\sigma}_1$. Then $\sigma_1 = h(\sigma_1) = \min\{h(\sigma_1), 1\} = \hat{\sigma}^*((\sigma_1, 1))$, where we have used the fact that $\tilde{\sigma}_1 < 1$.

The converses are now immediate. ■

So how can we use Figure 2 to think about FP equilibrium? Consider a symmetric strategy profile $\sigma = (\sigma_1, 1) \in \Gamma^{\text{FP}}$. Since $\sigma_1 < 1$ it corresponds to a response that cannot be optimal when the point $(\underline{\pi}_1, \bar{\pi}_1)$ lies strictly within the blue triangle. By Lemma 3.2, the location of σ_1 relative to $\tilde{\sigma}$ determines the region within Figure 1 for which σ_1 is compatible with optimality: if $\sigma_1 < \tilde{\sigma}_1$ this is the green triangle, including its boundaries; if $\sigma_1 > \tilde{\sigma}_1$ this is the lower boundary of the pink rectangle (recalling that $\tilde{\sigma}_1 > \hat{\sigma}_1$ so $\hat{\sigma}^*((\sigma_1, 1)) < 1$ when $\sigma_1 > \tilde{\sigma}_1$); and if $\sigma_1 = \tilde{\sigma}_1$ this is the pink rectangle, including its boundaries. Hence, for each σ_1 we can identify conditions on $(\underline{\pi}_1, \bar{\pi}_1)$ which are necessary and sufficient for the optimality of σ_1 .

Figure 3 adds this information to Figure 2. For each σ_1 value the rationalising conditions on $(\underline{\pi}_1, \bar{\pi}_1)$ are indicated in green below the horizontal axis. The conditions are the same for

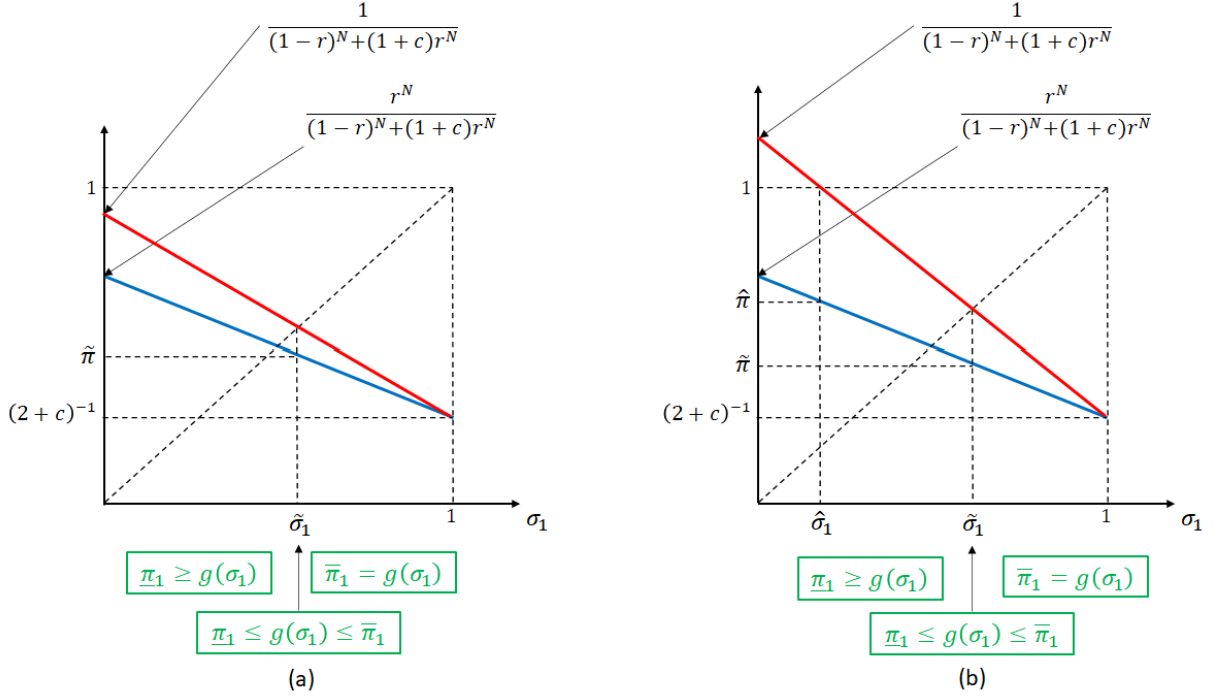


Figure 3: Rationalising σ_1

each panel:⁶ when $\sigma_1 < \tilde{\sigma}_1$ we need $\underline{\pi}_1 \geq g(\sigma_1)$; when $\sigma_1 = \tilde{\sigma}_1$ we need $\underline{\pi}_1 \leq g(\sigma_1) \leq \bar{\pi}_1$; and when $\sigma_1 > \tilde{\sigma}_1$ we need $\bar{\pi}_1 = g(\sigma_1)$. Most of the results in the following section can be understood by careful inspection of Figure 3, but we include non-graphical arguments for completeness.

3.1 Characterising the FP equilibrium

Here we characterise the value of σ_1 in an FP equilibrium, assuming one exists. In the following section we address the existence question. The following summarises some key features of Figure 3.

Lemma 3.3 *Suppose $(\sigma_1, 1) \in \Gamma^{\text{FP}}$ is an equilibrium. Then:*

1. $\sigma_1 < \tilde{\sigma}_1$ iff $\underline{\pi}_1 \geq g(\sigma_1) > \tilde{\pi}$.
2. $\sigma_1 > \tilde{\sigma}_1$ iff $\bar{\pi}_1 = g(\sigma_1) < \tilde{\pi}$.

⁶Condition (7), it turns out, is only relevant for the equilibrium existence conditions (see Section 3.2); it is irrelevant for the characterisation of the FP equilibrium when it exists.

3. $\sigma_1 = \tilde{\sigma}_1$ iff $\tilde{\pi} \in [\underline{\pi}_1, \bar{\pi}_1]$

Proof. Since $(\sigma_1, 1) \in \Gamma^{\text{FP}}$ we have $0 \leq \sigma_1 < 1$.

Suppose $\sigma_1 < \tilde{\sigma}_1$. Then $\sigma_1 < \hat{\sigma}^*((\sigma_1, 1))$ by Lemma 3.2. From Figure 1 we deduce that $\underline{\pi}_1 \geq \pi^*((\sigma_1, 1)) = g(\sigma_1) > g(\tilde{\sigma}_1) = \tilde{\pi}$.

Suppose $\sigma_1 > \tilde{\sigma}_1$. Then $\sigma_1 > \hat{\sigma}^*((\sigma_1, 1))$ by Lemma 3.2. Since $\sigma_1 < 1$ we deduce from Figure 1 that $\bar{\pi}_1 = \pi^*((\sigma_1, 1)) = g(\sigma_1) < g(\tilde{\sigma}_1) = \tilde{\pi}$.

Suppose $\sigma_1 = \tilde{\sigma}_1$. Then $\sigma_1 = \hat{\sigma}^*((\sigma_1, 1))$ by Lemma 3.2. From Figure 1 we deduce that $\tilde{\pi} = \pi^*((\tilde{\sigma}_1, 1)) \in [\underline{\pi}_1, \bar{\pi}_1]$.

The converses are now immediate. ■

It is now straightforward to characterise the FP equilibrium value of σ_1 . Suppose, for example, that $\underline{\pi}_1 > \tilde{\pi}$. Then σ_1 must satisfy $\sigma_1 < \tilde{\sigma}_1$ and $\underline{\pi}_1 \geq g(\sigma_1) > \tilde{\pi}$ (see Figure 3 or Lemma 3.3). If $\sigma_1 \in (0, \tilde{\sigma}_1)$ then it can only be rationalised if $\underline{\pi}_1 = g(\sigma_1)$ – see Figure 1 – so we must have $\sigma_1 = g^{-1}(\underline{\pi}_1)$; conversely, if there is no $x \in (0, \tilde{\sigma})$ with $\underline{\pi}_1 = g(x)$ – that is, if $\underline{\pi}_1 > g(0)$ – then we can only rationalise $\sigma_1 = 0$.

Corollary 3.1 *Suppose $(\sigma_1, 1) \in \Gamma^{\text{FP}}$ is an equilibrium.*

1. *If $\underline{\pi}_1 > \tilde{\pi}$ then:*

(a) $\sigma_1 = g^{-1}(\underline{\pi}_1)$ if $\underline{\pi}_1 \leq g(0)$; and

(b) $\sigma_1 = 0$ if $\underline{\pi}_1 > g(0)$.

2. *If $\bar{\pi}_1 < \tilde{\pi}$ then $\sigma_1 = g^{-1}(\bar{\pi}_1)$.*

3. *If $\tilde{\pi} \in [\underline{\pi}_1, \bar{\pi}_1]$ then $\sigma_1 = \tilde{\sigma}_1$.*

Proof. Since $(\sigma_1, 1) \in \Gamma^{\text{FP}}$ we have $0 \leq \sigma_1 < 1$.

Suppose $\underline{\pi}_1 > \tilde{\pi}$. Then from Lemmas 3.2-3.3 we deduce that $\underline{\pi}_1 \geq g(\sigma_1)$ and $\sigma_1 < \hat{\sigma}^*((\sigma_1, 1))$. If $\underline{\pi}_1 < g(0)$ then $g(0) > g(\sigma_1)$ and therefore $\sigma_1 > 0$. Given $\sigma_1 < \hat{\sigma}^*((\sigma_1, 1))$, Figure 1 implies

$$\sigma_1 > 0 \quad \Leftrightarrow \quad \underline{\pi}_1 = \pi^*((\sigma_1, 1)) = g(\sigma_1) \quad \Leftrightarrow \quad \sigma_1 = g^{-1}(\underline{\pi}_1).$$

If $\underline{\pi}_1 \geq g(0)$ then $\underline{\pi}_1 > g(x)$ for any $x > 0$ so $\sigma_1 = 0$.

If $\bar{\pi}_1 < \tilde{\pi}$ then Lemmas 3.2-3.3 give $\bar{\pi}_1 = g(\sigma_1)$ and $\sigma_1 > \hat{\sigma}^*((\sigma_1, 1))$. Hence:

$$\sigma_1 = g^{-1}(\bar{\pi}_1) > \hat{\sigma}^*((\sigma_1, 1)).$$

(It follows that $\bar{\pi}_1 > g(1)$.)

If $\tilde{\pi} \in [\underline{\pi}_1, \bar{\pi}_1]$ then Lemmas 3.2-3.3 imply $\sigma_1 = \hat{\sigma}^*((1, \sigma_1))$. ■

Corollary 3.1 characterises the FP equilibrium, assuming such exists. This characterisation is in terms of the parameters $\underline{\pi}_1$ and $\bar{\pi}_1$ and the functions g and h (which in turn depend on parameters r , c and N). Next, we identify necessary and sufficient conditions for our existence assumption to be warranted.

Note that if $\tilde{\pi}$ lies above the interval $[\underline{\pi}_1, \bar{\pi}_1]$ then the upper limit $\bar{\pi}_1$ determines the value of σ_1 in equilibrium; and conversely, when $\tilde{\pi}$ is below $[\underline{\pi}_1, \bar{\pi}_1]$ it is the lower limit $\underline{\pi}_1$ that determines σ_1 .

3.2 Existence of an FP equilibrium

To help analyse the existence conditions for FP equilibrium, it will be useful to define the following set:

$$\Gamma = \{(x, y) \in [0, 1]^2 \mid 0 < x \leq y < 1\}.$$

Note that $(\underline{\pi}_t, \bar{\pi}_t) \in \Gamma$ for each $t \in \{1, 2\}$.

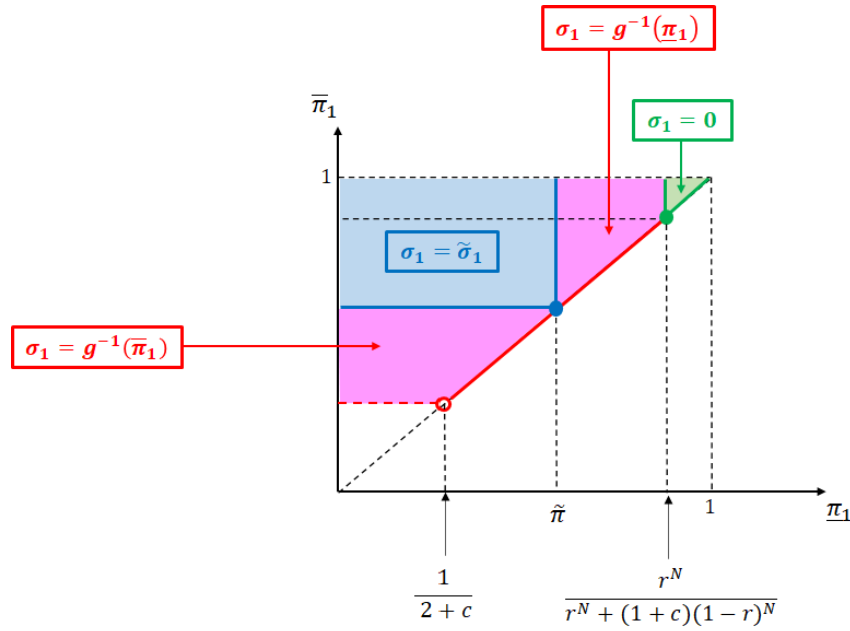


Figure 4: FP equilibrium value for σ_1

If $(\sigma_1, 1) \in \Gamma^{\text{FP}}$ is an equilibrium then Corollary 3.1 implies:

$$\sigma_1 = \begin{cases} 0 & \text{if } \underline{\pi}_1 > g(0) \\ g^{-1}(\underline{\pi}_1) & \text{if } \tilde{\pi} < \underline{\pi}_1 \leq g(0) \\ \tilde{\sigma}_1 & \text{if } \underline{\pi}_1 \leq \tilde{\pi} \leq \bar{\pi}_1 \\ g^{-1}(\bar{\pi}_1) & \text{if } \bar{\pi}_1 < \tilde{\pi} \end{cases} \quad (8)$$

Figure 4 illustrates. Conversely, σ_1 is well-defined by (8), and lies in $[0, 1)$, for any $(\underline{\pi}_1, \bar{\pi}_1) \in \Gamma$ with $\bar{\pi}_1 > g(1)$. However, if $\bar{\pi}_1 \leq g(1)$ then there is no $\sigma_1 \in [0, 1)$ such that $g(\sigma_1) = \bar{\pi}_1$ so no FP equilibrium can exist. Hence,

$$\bar{\pi}_1 > g(1) \quad (9)$$

is a necessary condition for the existence of an FP equilibrium. In particular, if $\underline{p} = \bar{p}$ so that $\underline{\pi}_1 = \bar{\pi}_1 (\equiv \pi_1)$ then we recover the specification of Feddersen and Pesendorfer's (1998) responsive equilibrium:

$$\sigma_1 = \begin{cases} 0 & \text{if } \pi_1 > g(0) \\ g^{-1}(\pi_1) & \text{if } g(1) < \pi_1 \leq g(0) \end{cases}$$

It remains to add a condition that is necessary and sufficient to rationalise $\sigma_2 = 1$. The appropriate condition depends on whether $h(\sigma_1) \geq 1$ (hence $\hat{\sigma}^*(\sigma) = 1$) or $h(\sigma_1) < 1$ (so $\hat{\sigma}^*(\sigma) < 1$). In the former case, rationalisation requires $\underline{\pi}_2 \leq g(\sigma_1)$ and in the latter $\bar{\pi}_2 \leq g(\sigma_1)$: see Figure 1 and recall that $g(\sigma_1) = \pi^*((\sigma_1, 1))$. Of course, these rationalisation conditions involve an endogenous variable so are not suitable in their present form. To transform them into conditions on exogenous parameters we distinguish two cases: voting problems for which (7) holds and those for which it fails.

3.2.1 Voting problems that satisfy (7)

Suppose $V \in \mathcal{V}$ satisfies (7). Then $h(\sigma_1) < 1$, and hence $\hat{\sigma}^*(\sigma) < 1$, for all $\sigma \in \Gamma^{\text{FP}}$. In this case, the optimality of $\sigma_2 = 1$ is equivalent to $\bar{\pi}_2 \leq g(\sigma_1)$.

Proposition 3.1 *If $V \in \mathcal{V}$ satisfies (7) then there exists an FP equilibrium iff one of the following conditions holds:*

$$g(1) < \bar{\pi}_1 < \tilde{\pi} \quad (i)$$

$$\max\{\underline{\pi}_1, \bar{\pi}_2\} \leq \tilde{\pi} \leq \bar{\pi}_1 \quad (ii)$$

$$\tilde{\pi} < \underline{\pi}_1 \text{ and } \bar{\pi}_2 \leq \min\{\underline{\pi}_1, g(0)\} \quad (iii)$$

Proof. If $\underline{\pi}_1 < \tilde{\pi}$ then (8) implies $g(\sigma_1) = \underline{\pi}_1 > \bar{\pi}_2$. Hence, (9) is necessary and sufficient for existence of an FP equilibrium when $\underline{\pi}_1 < \tilde{\pi}$. This gives condition (i).

If $\underline{\pi}_1 \leq \tilde{\pi} \leq \bar{\pi}_1$ then (9) is satisfied and $g(\sigma_1) = \tilde{\pi}$ in an FP equilibrium. The latter means that $\bar{\pi}_2 \leq g(\sigma_1)$ is equivalent to $\bar{\pi}_2 \leq \tilde{\pi}$. This gives condition (ii).

The final two components of (8) generate condition (iii). If $\tilde{\pi} < \underline{\pi}_1 \leq g(0)$ then (9) is satisfied and (8) implies $g(\sigma_1) = \underline{\pi}_1$. Hence, $\bar{\pi}_2 \leq g(\sigma_1)$ is equivalent to $\bar{\pi}_2 \leq \underline{\pi}_1$ and we have the following condition:

$$\tilde{\pi} < \underline{\pi}_1 < g(0) \text{ and } \underline{\pi}_1 \geq \bar{\pi}_2 \quad (10)$$

Finally, if $\underline{\pi}_1 > g(0)$ then $\sigma_1 = 0$ in an FP equilibrium so $\bar{\pi}_2 \leq g(\sigma_1)$ is equivalent to $\bar{\pi}_2 \leq g(0)$, giving condition:

$$\bar{\pi}_2 \leq g(0) \leq \underline{\pi}_1 \quad (11)$$

Conditions (10)-(11) are jointly equivalent to (iii). ■

When reading the conditions in Proposition 3.1 recall that

$$g(0) = \frac{r^N}{(1-r)^N + (1+c)r^N}$$

$$g(1) = (2+c)^{-1}$$

and $\tilde{\pi}$ is an implicitly defined function of the model parameters:

$$\tilde{\pi} = \tilde{\sigma}_1 [(1-r)\tilde{\sigma}_1 + r]^N$$

where $\tilde{\sigma}_1$ is the fixed point of h .

It is not hard to see that we may re-express conditions (i)-(iii) in the following more compact form:

Corollary 3.2 *If $V \in \mathcal{V}$ satisfies (7) then an FP equilibrium exists iff $\bar{\pi}_1 > g(1)$, $\underline{\pi}_2 \leq g(0)$ and one of the following conditions holds:*

$$\underline{\pi}_1 < \bar{\pi}_2 \leq \tilde{\pi} \quad (\text{I})$$

$$\underline{\pi}_1 \geq \bar{\pi}_2 \quad (\text{II})$$

3.2.2 Voting problems that do not satisfy (7)

When $V \in \mathcal{V}$ does not satisfy (7) the analysis leading to condition (iii) of Proposition 3.1 must be amended. When $\sigma_1 \leq \hat{\sigma}_1$ we have $\hat{\sigma}^*((\sigma_1, 1)) = 1$ so rationalising $\sigma_2 = 1$ requires only $\underline{\pi}_2 \leq g(\sigma_1)$, rather than the more stringent condition $\bar{\pi}_2 \leq g(\sigma_1)$. Note that $\sigma_1 \leq \hat{\sigma}_1$ is equivalent to $g(\sigma_1) \geq \hat{\pi}$.

Proposition 3.2 *If $V \in \mathcal{V}$ does not satisfy (7) then there exists an FP equilibrium iff one of the following conditions holds:*

$$g(1) < \bar{\pi}_1 < \tilde{\pi} \quad (\text{i})$$

$$\max\{\underline{\pi}_1, \bar{\pi}_2\} \leq \tilde{\pi} \leq \bar{\pi}_1 \quad (\text{ii})$$

$$\tilde{\pi} < \underline{\pi}_1 \text{ and } \underline{\pi}_2 \leq g(0) \text{ and } [\bar{\pi}_2 \leq \underline{\pi}_1 \text{ or } \hat{\pi} \leq \underline{\pi}_1] \quad (\text{iii}')$$

Proof. The analysis leading to conditions (i) and (ii) is the same as for the proof of Proposition 3.1.

If $\tilde{\pi} < \underline{\pi}_1 \leq g(0)$ then (8) implies $g(\sigma_1) = \underline{\pi}_1$ so $\underline{\pi}_2 \leq g(\sigma_1)$ is equivalent to $\underline{\pi}_2 \leq \underline{\pi}_1$ which is always satisfied. It follows that condition (10) is necessary to rationalise $\sigma_2 = 1$ only if $\underline{\pi}_1 < \hat{\pi}$. Recalling that $\hat{\pi} \leq g(0)$ by Lemma 3.1 we therefore have conditions

$$\tilde{\pi} < \underline{\pi}_1 < \hat{\pi} \text{ and } \underline{\pi}_1 \geq \bar{\pi}_2 \quad (12)$$

$$\hat{\pi} \leq \underline{\pi}_1 \leq g(0) \quad (13)$$

If $\underline{\pi}_1 \geq g(0)$ then $\underline{\pi}_1 \geq \hat{\pi}$ and we have $\sigma_1 = 0$ in an FP equilibrium. The necessary and sufficient condition to rationalise $\sigma_2 = 1$ is therefore $\underline{\pi}_2 \leq g(0)$. This gives condition:

$$\underline{\pi}_2 \leq g(0) \leq \underline{\pi}_1 \quad (14)$$

Satisfying one of conditions (12)-(14) is equivalent to satisfying condition (iii'). ■

Comparing Propositions 3.1 and 3.2 we see that condition (iii') in the latter relaxes condition (iii) in the former, but this is the only change. Once again, we may re-express the conditions of Proposition 3.2 in a more compact form:

Corollary 3.3 *If $V \in \mathcal{V}$ does not satisfy (7) then there exists an FP equilibrium iff $\bar{\pi}_1 > g(1)$, $\underline{\pi}_2 \leq g(0)$ and one of the following conditions holds:*

$$\underline{\pi}_1 < \bar{\pi}_2 \leq \tilde{\pi} \quad (\text{I})$$

$$\underline{\pi}_1 \geq \min\{\bar{\pi}_2, \hat{\pi}\} \quad (\text{II}')$$

Condition (II') relaxes condition (II) from Corollary 3.2.

4 Dual FP equilibria

Consider responsive equilibria with $0 = \sigma_1 < \sigma_2 < 1$. These are, in a natural sense, “dual” to the FP equilibria. Let $\Gamma^{\text{DFP}} = \{(0, \sigma_2) \mid 0 < \sigma_2 < 1\}$. Ryan (2021, Lemma 4.4) establishes that Γ^{DFP} contains no equilibria when N is sufficiently large but has nothing more to say about this class of strategies.

The members of Γ^{DFP} share the property that a voter is only pivotal if all other voters receive the guilty signal. In the absence of ambiguity, a profile in Γ^{DFP} is an equilibrium only if voters are indifferent between acquittal and conviction conditional on knowing that all $N + 1$ signals are guilty signals. This somewhat implausible – and uninteresting – scenario is often excluded by assumption: parameter restrictions are imposed to ensure that conviction is strictly preferred to acquittal conditional on $N + 1$ guilty signals. When $\underline{p} = \bar{p} = p$ this assumption is:

$$p < \frac{r^{N+1}}{r^{N+1} + (1+c)(1-r)^{N+1}} \quad (\eta_0)$$

Equivalently, if π_b is the posterior after conditioning on one’s own guilty signal, then the assumption becomes:

$$\pi_2 < \frac{r^N}{r^N + (1+c)(1-r)^N} \quad (\eta_1)$$

In the presence of ambiguity, matters are not quite so straightforward. One might think that a sufficient condition for excluding an equilibrium in Γ^{DFP} would be to impose assumption η_0 with $p = \bar{p}$ (equivalently, η_1 with $\pi_2 = \bar{\pi}_2$). This implies that η_0 holds for *any* $p \in [\underline{p}, \bar{p}]$. Consider a voter responding to some $\sigma \in \Gamma^{\text{DFP}}$ who receives a guilty signal. The option of voting to acquit (which guarantees acquittal under the unanimity rule) would be evaluated using the lowest possible prior, \underline{p} , as this maximises the probability of a wrong decision. If the option of voting to convict is likewise evaluated using \underline{p} it follows that all responses will be evaluated using \underline{p} so the voter strictly prefers voting for conviction than for acquittal, just as in the case without ambiguity. They will not randomise. But if the option of voting to convict has a lower expected utility when evaluated using \bar{p} rather than \underline{p} , then it will be evaluated using \bar{p} and we cannot, *a priori*, exclude the possibility that the voter is willing to randomise.

The analysis of Sections 3.1-3.2 is easily adapted to the task of characterising the dual FP equilibria and the conditions for their existence. The details are relegated to the Appendix, where the following is proved.⁷

⁷Proposition 4.1 confirms that there is no dual FP equilibrium when condition η_1 holds with $\pi_2 = \bar{\pi}_2$. The latter condition will be satisfied if N is large enough, so dual FP equilibria can only exist when juries are “small” (Ryan, 2021, Lemma 4.4).

Proposition 4.1 *If $\sigma \in \Gamma^{\text{DFP}}$ then*

$$\pi^*(\sigma) = g(0) \equiv \frac{r^N}{(1-r)^N + (1+c)r^N}.$$

There exists an equilibrium in Γ^{DFP} iff

(i) $\underline{\pi}_2 = g(0)$; or

(ii) $\underline{\pi}_2 < g(0) \leq \min\{\underline{\pi}_1, \bar{\pi}_2\}$ and $h(0) < 1$.

If (i) holds then $\sigma \in \Gamma^{\text{DFP}}$ is an equilibrium iff

$$\sigma_2 \leq h(0)^{\frac{1}{N+1}} \tag{15}$$

If (ii) holds then $\sigma \in \Gamma^{\text{DFP}}$ is an equilibrium iff

$$\bar{\pi}_2 = g(0) \quad \text{and} \quad \sigma_2 \geq h(0)^{\frac{1}{N+1}} \tag{16}$$

or

$$\underline{\pi}_2 < g(0) < \bar{\pi}_2 \quad \text{and} \quad \sigma_2 = h(0)^{\frac{1}{N+1}} \tag{17}$$

Only equilibria in category (i) exist in the absence of ambiguity. Such equilibria are “non-generic”, requiring a voter to be indifferent between the conviction and acquittal outcomes conditional on knowing that all $N + 1$ signals are guilty signals. Equilibria in category (ii), on the other hand, are generic but can only arise when non-trivial ambiguity is present.

5 Concluding remarks

Combining the results of Ryan (2021) with those of the present paper, one can identify *all* the symmetric equilibria of *any* voting problem. A complex picture emerges.

Figure 3 of Ryan (2021) already shows that some voting problems have multiple non-trivial equilibria, including cases where MNR and SMR equilibria co-exist. The existence conditions for FP and DFP equilibria do not fit neatly into this figure since restrictions on $\bar{\pi}_1$ and $\underline{\pi}_2$ are also involved. However, it is easy to see that many combinations of equilibria are possible. For example, FP and DFP equilibria may co-exist (e.g., for voting problems with $h(0) < 1$ and $\underline{\pi}_1 \geq \bar{\pi}_2 \geq g(0) \geq \underline{\pi}_2$). We may also observe voting problems with equilibria from the FP, MNR and SMR categories. To see that this is possible, recall that voting problems from region B in Figure 3 of Ryan (2021) possess both MNR and SMR equilibria. There are voting problems from the lower part of region B that also possess an FP equilibrium (e.g., those with $\bar{\pi}_2 \leq \tilde{\pi}$ and $\bar{\pi}_1 > g(1) = (2+c)^{-1}$). Likewise, there exist

voting problems in the upper portion of the red area of Figure 3 with equilibria from both the MNR and DFP classes.

In short, adding ambiguity to prior beliefs substantially complicates the equilibrium landscape. Without ambiguity we have the generic absence of multiple non-trivial equilibria and the generic confinement of non-trivial equilibria to the C and FP classes. When ambiguity is present: (i) non-trivial equilibria may take a variety of forms; (ii) there is pervasive multiplicity of non-trivial equilibria; and (iii) there exist generic voting problems with non-trivial equilibria, all of which are outside the C and FP categories. We are currently exploring the implications of ambiguity on decision quality, both theoretically and experimentally. Results will appear in a subsequent paper.

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Appendix

In this Appendix we analyse the so-called “dual” FP equilibria and prove Proposition 4.1.

First, using (1)-(2) and (4) we observe that $\pi^*((0, \sigma_2)) = g(0)$ for any $\sigma_2 \in (0, 1)$, which establishes the first claim in Proposition 4.1.

Now define the function $\bar{h} : (0, 1] \rightarrow \mathbb{R}_+$ as follows:

$$\bar{h}(\sigma_2) \equiv \sigma^*((0, \sigma_2)) = \frac{1}{\left[r^N + (1+c)(1-r)^N \right] \sigma_2^N}.$$

Thus, for any $\sigma \in \Gamma^{\text{DFP}}$ we have $\bar{h}(\sigma_2) > \pi^*((0, \sigma_2))$. Moreover, $\lim_{\sigma_2 \rightarrow 0} \bar{h}(\sigma_2) = \infty$ and \bar{h} is strictly decreasing with $\bar{h}(1) < 1$ iff (7) holds.⁸ Thus, if (and only if) the latter condition holds there exist $\tilde{\sigma}_2, \hat{\sigma}_2 \in (0, 1)$ such that $\bar{h}(\tilde{\sigma}_2) = \tilde{\sigma}_2$ and $\bar{h}(\hat{\sigma}_2) = 1$. Both are obviously unique, with $\tilde{\sigma}_2 > \hat{\sigma}_2$. Indeed, direct computation gives:

$$\tilde{\sigma}_2 = \left[\frac{1}{r^N + (1+c)(1-r)^N} \right]^{\frac{1}{N+1}} = h(0)^{\frac{1}{N+1}}$$

and

$$\hat{\sigma}_2 = \tilde{\sigma}_2^{\frac{N+1}{N}}.$$

If (7) does *not* hold, then $\hat{\sigma}^*((0, \sigma_2)) = 1$ for all $\sigma_2 \in (0, 1)$. By inspection of Figure 1 we have:

- If (7) does *not* hold, then $\sigma \in \Gamma^{\text{DFP}}$ is an equilibrium iff $\underline{\pi}_2 = g(0)$ (since this implies $\underline{\pi}_1 > g(0)$).

The following may be established by the same argument used to prove Lemma 3.2 applied to \bar{h} (*mutatis mutandis*):

Lemma 5.1 *Suppose $(0, \sigma_2) \in \Gamma^{\text{DFP}}$ and (7) holds. Then $\sigma_2 \geq \tilde{\sigma}_2$ iff $\sigma_2 \geq \hat{\sigma}^*((0, \sigma_2))$.*

Using Lemma 5.1 and Figure 1 we now deduce:

- If (7) holds and $\sigma_2 < \tilde{\sigma}_2$ then $(0, \sigma_2) \in \Gamma^{\text{DFP}}$ is an equilibrium iff $\underline{\pi}_2 = g(0)$.
- If (7) holds and $\sigma_2 > \tilde{\sigma}_2$ then $(0, \sigma_2) \in \Gamma^{\text{DFP}}$ is an equilibrium iff $\bar{\pi}_2 = g(0) \leq \underline{\pi}_1$.
- If (7) holds and $\sigma_2 = \tilde{\sigma}_2$ then $(0, \sigma_2) \in \Gamma^{\text{DFP}}$ is an equilibrium iff $\underline{\pi}_2 < g(0) < \bar{\pi}_2$ and $g(0) \leq \underline{\pi}_1$.

The remaining claims in Proposition 4.1 follow by collecting the facts in the bullet points above.

⁸Note that condition (7) may be written: $h(0) < 1$. This is the form in which it appears in Proposition 4.1.